MATH3280A Introductory Probability, 2014-2015
Solutions to HW1

P.34 Ex.8
Let A_1, A_2, ..., A_n be n events. Show that if
$P(A_1) = P(A_2) = ... = P(A_n) = 1$, then $P(A_1A_2...A_n) = 1$.

Solution
For any $i \in \{1, 2, ..., n\}, P(A_i^c) = 1 - P(A_i) = 0$.
Then we have

$$\begin{aligned} P((A_1A_2...A_n)^c) &= P(A_1^c \cup A_2^c \cup ... \cup A_n^c) \\ &\leq \sum_{i=1}^{n} P(A_i^c) \quad \text{(by countable subadditivity of P)} \\ &= 0. \end{aligned}$$

Hence $P((A_1A_2...A_n)^c) = 0$ and $P(A_1A_2...A_n) = 1 - P((A_1A_2...A_n)^c) = 1$.
□

P.35 Ex.12
(Borel-Cantelli lemma)
Let $A_1, A_2, A_3, ...$ be a sequence of events.
Prove that if the series $\sum\limits_{n=1}^{\infty} P(A_n)$ converges, then $P(\bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=m}^{\infty} A_n) = 0$.

Solution
Let $B_m = \bigcup\limits_{n=m}^{\infty} A_n$ for $m = 1, 2, ...$
Then $\{B_m\}_{m=1}^{\infty}$ is a decreasing sequence of events.

$$\begin{aligned} P(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n) &= P(\bigcap_{m=1}^{\infty} B_m) \\ &= \lim_{m} P(B_m) \\ &= \lim_{m} P(\bigcup_{n=m}^{\infty} A_n) \\ &\leq \lim_{m} \sum_{n=m}^{\infty} P(A_n). \quad \text{(countable subadditivity)} \end{aligned}$$

The tail of a convergent series tends to zero:
Let $L = \sum\limits_{n=1}^{\infty} P(A_n) < \infty$. We have

$$\begin{aligned} \lim_{m} \sum_{n=m}^{\infty} P(A_n) &= \lim_{m} \Big(\lim_{k} \sum_{n=m}^{k} P(A_n) \Big) \\ &= \lim_{m} \Big(\lim_{k} \sum_{n=1}^{k} P(A_n) - \sum_{n=1}^{m-1} P(A_n) \Big) \\ &= \lim_{k} \sum_{n=1}^{k} P(A_n) - \lim_{m} \sum_{n=1}^{m-1} P(A_n) \\ &= L - L \\ &= 0. \end{aligned}$$

Hence

$$P(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n) = 0.$$

□